

# Stochastic Optimal Control of Pairs Trading Strategies with Absolute and Relative Inventory Penalties<sup>☆</sup>

Ali Al-Aradi

*Department of Statistical Sciences, University of Toronto, Toronto, Canada*

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## Abstract

In this paper, we apply techniques from stochastic control theory to derive the optimal trading rules for a pair of cointegrated assets.

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## 1. Introduction

Statistical arbitrage (StatArb) trading strategies are a class of algorithmic trading strategies predicated on the notion that the movements of portfolios of multiple assets can be more accurately predicted than those of individual assets. Strategies of this sort attempt to exploit the mean-reverting properties of certain portfolios of assets to make a profit. StatArb strategies originally evolved from simpler pairs trading strategies, which involve two assets whose joint behavior displayed a mean-reverting quality. Intuitively, pairs trading strategies involve betting on the relative values of the two assets; when the asset prices begin to diverge and the spread between the two assets becomes too large, a long (resp. short) position is taken in the underpriced (resp. overpriced) asset.

In this paper, we apply techniques from stochastic control theory to derive the optimal trading rules for the two assets. The agent trades in the two assets over a fixed investment horizon, and then liquidates her position at the terminal date with the objective of maximizing the expected proceeds. Additionally, we assume the presence of temporary price impacts which force the agent to complete her transactions at less favorable prices when increasing her trading activity. Finally, the agent also incorporates two non-financial penalties into her objective function: absolute and relative inventory running penalties. An absolute inventory penalty is equivalent to setting a target risk level since a smaller absolute position in the two assets translates into a lower level of total risk. This penalty dissuades the agent from

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remaining in a large long or short position in either asset for too long. A relative position penalty can be viewed as a way of forcing the strategy to be self-financing; a trading rule that forces the agent to inject additional cash in the strategy (or hold a surplus of cash for too long) is penalized.

## 2. Model Setup

### 2.1. Asset Price Dynamics

We assume that the agent can invest in two assets whose price at time  $t$  is given by  $S_t^1$  and  $S_t^2$ , respectively. Furthermore, we assume that these two assets are cointegrated so that their “price half-spread,”  $\varepsilon$ , is mean-reverting. This is achieved by modeling  $\varepsilon$  using an Ornstein-Uhlenbeck process:

$$\varepsilon_t = \frac{1}{2} (S_t^1 - S_t^2) \tag{1}$$

$$d\varepsilon_t = -\kappa \varepsilon_t dt + \sigma dW_t^1 \tag{2}$$

We also assume that there is a martingale component,  $\ell$ , defined as follows:

$$\ell_t = \frac{1}{2} (S_t^1 + S_t^2) \tag{3}$$

$$d\ell_t = \eta dW_t^2 \tag{4}$$

where  $W_t^1$  and  $W_t^2$  are standard Wiener processes with instantaneous correlation given by  $\rho$ , i.e.  $d[W^1, W^2]_t = \rho dt$ . The specification above implies that the asset prices are given by:

$$S_t^1 = \ell_t + \varepsilon_t \tag{5}$$

$$S_t^2 = \ell_t - \varepsilon_t \tag{6}$$

Based on this model, the asset prices satisfy the following dynamics:

$$dS_t^1 = -\frac{\kappa}{2}(S_t^1 - S_t^2) dt + \sigma dW_t^1 + \eta dW_t^2 \tag{7}$$

$$dS_t^2 = \frac{\kappa}{2}(S_t^1 - S_t^2) dt - \sigma dW_t^1 + \eta dW_t^2 \tag{8}$$

The drift terms demonstrate the mean-reverting property that the pairs trading strategy looks to exploit. When  $S_t^1 > S_t^2$ ,  $S^2$  has a positive drift term and  $S^1$  has a negative drift term, suggesting that the agent should enter into a long position in the former and a short position in the latter. The relationship is reversed in the case where  $S_t^1 < S_t^2$ . These two conditions can be written in terms of the cointegrating factor; the first corresponds to the case where  $\varepsilon_t > 0$  and the second corresponds to  $\varepsilon_t < 0$ . Thus,  $\varepsilon$  is the indicator of a relative mispricing

in the two assets. Furthermore, from these SDEs we can also deduce that the instantaneous volatilities of the two assets are  $V_1 = \sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho}$  and  $V_2 = \sqrt{\sigma^2 + \eta^2 - 2\sigma\eta\rho}$ , and the instantaneous covariance (i.e. the quadratic covariation) is  $C_{12} = \eta^2 - \sigma^2$ . It should also be noted that this model does not guarantee positive stock prices, though this can be ignored if we assume that investment horizons are sufficiently short.

In our specification, the fact that the cointegrating factor's mean-reversion level is zero implies that the price of the two assets will converge to the same value in the long run. More generally, however, StatArb strategies involve a cointegration factor of the form  $\varepsilon_t = AS_t^1 + BS_t^2$ , where  $A$  and  $B$  must be estimated, and the mean reversion level would be some value,  $\theta$ , which is not necessarily zero as is the case in this specification. This would alter the level at which the assets would be considered over/underpriced relative to one another, but it does not materially affect the solution to the control problem, and so we exclude it for simplicity. Additionally, the specification given above assumes that neither asset has a long-term drift. To include drift to the two assets the auxiliary martingale component,  $\ell$ , would require a drift term which would then be inherited by the two assets. Once again, this does not have a significant impact on the solution methodology and is ignored.

## 2.2. Trading Dynamics

The agent trades continuously in the two assets at rate  $\nu_t$  in  $S^1$  and  $\mu_t$  in  $S^2$  (which may be negative). Denoting the holdings in the two assets as  $\alpha_t$  and  $\beta_t$  in  $S^1$  and  $S^2$ , respectively, the holdings evolve through time according to the following dynamics:

$$d\alpha_t = \nu_t dt \tag{9}$$

$$d\beta_t = \mu_t dt \tag{10}$$

We also assume that the extent of the agent's trading has an impact on the execution price. This is modeled using a linear temporary price impact function, so that execution prices in the two assets are given by:

$$\widehat{S}_t^1 = S_t^1 + a\nu_t \tag{11}$$

$$\widehat{S}_t^2 = S_t^2 + b\mu_t \tag{12}$$

We also need to keep track of the agent's cash holdings. This will depend on the rate of trading in both assets along with the execution price the agent secures through time:

$$dX_t = -\widehat{S}_t^1 \nu_t dt - \widehat{S}_t^2 \mu_t dt; \quad X_0 = 0 \tag{13}$$

Note that the negative signs indicate that the agent is buying shares and therefore her cash holdings are decreasing. Using the definitions for the execution prices, we can rewrite the

cash process as a function of  $\varepsilon$  and  $\ell$ , which will come into play when solving the optimal control problem:

$$\begin{aligned} \Rightarrow X_T &= - \int_0^T (S_t^1 + a\nu_t) \cdot \nu_t dt - \int_0^T (S_t^2 + b\mu_t) \cdot \mu_t dt \\ &= - \int_0^T (\ell_t + \varepsilon_t + a\nu_t) \cdot \nu_t dt - \int_0^T (\ell_t - \varepsilon_t + b\mu_t) \cdot \mu_t dt \end{aligned} \quad (14)$$

### 3. Stochastic Optimal Control Problem

We now focus on setting up the agent's optimal control problem. The agent's primary objective is to maximize her profits over the investment horizon, while minimizing some non-financial inventory penalties. Her profits consist of her cash holdings on the terminal date,  $X_T$ , and the proceeds of liquidating the shares held in the two assets. We will assume a terminal liquidation penalty of  $c$  per share for both assets, so that the sale price is lower than the fundamental price and becomes less favorable the larger the agent's inventory at the terminal date. If the number of shares held is negative (i.e. the agent is short the asset), the liquidation price will be greater than the fundamental price, which is once again less favorable for the agent since she must buy back the shares at a higher price (which increases with inventory) to close her position.

To factor in the inventory-related running penalties we first use the sum of the square of the number of shares held for the absolute inventory penalty, i.e.  $\alpha_t^2 + \beta_t^2$ . This penalizes larger positions, both long and short, in either of the two assets and forces the agent to maintain a lower level of total risk consumption. One could use the sum of the absolute value of the number of shares held, but the square function is more analytically tractable. For the relative inventory penalty we use the squared sum of the number of shares held  $(\alpha_t + \beta_t)^2$ . In effect, this forces the agent to have holdings that are similar in magnitude but in the opposite direction, i.e.  $\alpha_t = -\beta_t$ . When the prices of the two assets are approximately equal and the investment horizon is short, this becomes similar to imposing a self-financing requirement, since the short-sale of one asset will (mostly) fund the purchase of the other. These assumptions about spread and horizon are reasonable if we believe that markets are efficient; in efficient markets the level of mispricing would be small and would close quickly.

The self-financing interpretation is only approximate because this formulation of the relative inventory running penalty will not ensure self-financing for the duration of the investment horizon. This is due to the fact that the prices of the two assets will drift and one share of Asset 1 will not be able to fund 1 share of Asset 2 once their prices diverge. The self-financing requirement can be made more rigid by modifying the penalty to  $(\gamma_t \alpha_t + \beta_t)^2$ , where  $\gamma_t$  is the ratio of the asset prices at time  $t$ .

Given the setup outlined above, we now give the full form of the agent's performance criteria,  $H^{(\nu, \mu)}$ , given trading rate processes  $(\nu, \mu)$  as follows:

$$H^{(\nu, \mu)}(t, \cdot) = \mathbb{E}_{t, \cdot} \left[ X_T + \underbrace{(S_T^1 - c\alpha_T) \cdot \alpha_T + (S_T^2 - c\beta_T) \cdot \beta_T}_{\text{liquidation costs}} - \underbrace{\phi \int_t^T (\alpha_s^2 + \beta_s^2) ds}_{\text{abs. position penalty}} - \underbrace{\psi \int_t^T (\alpha_s + \beta_s)^2 ds}_{\text{rel. position penalty}} \right] \quad (15)$$

Here,  $\phi$  and  $\psi$  are aversion parameters associated with the absolute and relative inventory running penalties, respectively. Increasing the value of these parameters increases the magnitude of the corresponding running penalty, forcing the agent to adjust accordingly.

The value function for the stochastic optimal control problem is defined as:

$$H(t, \cdot) = \sup_{\nu, \mu} H^{(\nu, \mu)}(t, \cdot) \quad (16)$$

and satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\left\{ \begin{array}{l} \partial_t H + \sup_{\nu, \mu} \left\{ \underbrace{\nu \cdot \partial_\alpha H + \mu \cdot \partial_\beta H + \frac{1}{2} \eta^2 \partial_{\ell\ell} H}_{\text{diffusion of } \alpha, \beta} - \underbrace{\kappa \varepsilon \cdot \partial_\varepsilon H + \frac{1}{2} \sigma^2 \cdot \partial_{\varepsilon\varepsilon} H + \frac{1}{2} \eta \sigma \rho \cdot \partial_{\ell\varepsilon} H}_{\text{diffusion of } \ell, \varepsilon; \mathcal{L}^{\ell, \varepsilon}} \right. \\ \left. - \underbrace{(\ell + \varepsilon + a\nu)\nu - (\ell - \varepsilon + b\mu)\mu - \phi(\alpha^2 + \beta^2) - \psi(\alpha + \beta)^2}_{\text{running rewards/penalties}} \right\} = 0 \\ \text{Terminal condition:} \quad H(T, \cdot) = \alpha(\ell + \varepsilon) + \beta(\ell - \varepsilon) - c(\alpha^2 + \beta^2) \end{array} \right. \quad (17)$$

Note that terminal cash,  $X_T$ , is viewed as a running reward term that appears in the HJB equation as opposed to a terminal reward term that appears in the terminal condition. Therefore, there is no need for  $x$  as a state variable; the only state variables needed are  $(\ell, \varepsilon, \alpha, \beta)$ .

## 4. Solving the HJB Equation

### 4.1. Case 1: $\phi > 0$ , $\psi > 0$

In this section we solve the HJB equation for the optimal trading rules in the two assets. To do so, we begin by rewriting the HJB equation, separating  $\nu$  and  $\mu$  terms in the supremum and completing the square:

$$\begin{aligned} & \partial_t H + \mathcal{L}^{\ell, \varepsilon} H - \phi(\alpha^2 + \beta^2) - \psi(\alpha + \beta)^2 \\ & + \sup_{\nu} \left\{ -a \left[ \nu - \frac{1}{2a} (\partial_\alpha H - (\ell + \varepsilon)) \right]^2 + \frac{1}{4a} (\partial_\alpha H - (\ell + \varepsilon))^2 \right\} \\ & + \sup_{\mu} \left\{ -b \left[ \mu - \frac{1}{2b} (\partial_\beta H - (\ell - \varepsilon)) \right]^2 + \frac{1}{4b} (\partial_\beta H - (\ell - \varepsilon))^2 \right\} = 0 \end{aligned}$$

From here we can read off the optimal controls (in feedback form) which are given by:

$$\nu^* = \frac{1}{2a} (\partial_\alpha H - (\ell + \varepsilon)) \quad (18)$$

$$\mu^* = \frac{1}{2b} (\partial_\beta H - (\ell - \varepsilon)) \quad (19)$$

Substituting these optimal controls back into the HJB equation we have:

$$\partial_t H + \mathcal{L}^{\ell, \varepsilon} H - (\phi + \psi)(\alpha^2 + \beta^2) - 2\psi\alpha\beta + \frac{1}{4a} (\partial_\alpha H - (\ell + \varepsilon))^2 + \frac{1}{4b} (\partial_\beta H - (\ell - \varepsilon))^2 = 0$$

Now, we propose the following ansatz for the value function based on the book value of the holdings and a remainder term:

$$H(t, \cdot) = \alpha(\ell + \varepsilon) + \beta(\ell - \varepsilon) + h(t, \varepsilon, \alpha, \beta)$$

This implies:

$$\begin{aligned} \partial_t H &= \partial_t h \\ \partial_\alpha H &= (\ell + \varepsilon) + \partial_\alpha h \\ \partial_\beta H &= (\ell - \varepsilon) + \partial_\beta h \\ \mathcal{L}^{\ell, \varepsilon} H &= \mathcal{L}^{\ell, \varepsilon} \{ \alpha(\ell + \varepsilon) + \beta(\ell - \varepsilon) \} + \mathcal{L}^{\ell, \varepsilon} h \\ &= -\kappa\varepsilon(\alpha - \beta) + \mathcal{L}^\varepsilon h \end{aligned}$$

$$\text{where } \mathcal{L}^\varepsilon = -\kappa\varepsilon \cdot \partial_\varepsilon + \frac{1}{2}\sigma^2 \cdot \partial_{\varepsilon\varepsilon}$$

Substituting these terms back into the HJB we have a new PDE in terms of  $h$ :

$$\begin{cases} (\partial_t + \mathcal{L}^\varepsilon)h + (\beta - \alpha)\kappa\varepsilon - (\phi + \psi)(\alpha^2 + \beta^2) - 2\psi\alpha\beta + \frac{1}{4a} (\partial_\alpha h)^2 + \frac{1}{4b} (\partial_\beta h)^2 = 0 \\ h(T, \cdot) = -c(\alpha^2 + \beta^2) \end{cases}$$

We proceed by proposing another ansatz of the following form:

$$h(t, \cdot) = f_2(t, \varepsilon) \cdot \alpha^2 + g_2(t, \varepsilon) \cdot \beta^2 + f_1(t, \varepsilon) \cdot \alpha + g_1(t, \varepsilon) \cdot \beta + u(t, \varepsilon) \cdot \alpha\beta + h_0(t, \varepsilon)$$

which implies:

$$\begin{aligned} (\partial_t + \mathcal{L}^\varepsilon)h &= (\partial_t + \mathcal{L}^\varepsilon) \left( f_2 \cdot \alpha^2 + g_2 \cdot \beta^2 + f_1 \cdot \alpha + g_1 \cdot \beta + u \cdot \alpha\beta + h_0 \right) \\ \partial_\alpha h &= 2f_2 \cdot \alpha + f_1 + u \cdot \beta \\ \partial_\beta h &= 2g_2 \cdot \beta + g_1 + u \cdot \alpha \\ \Rightarrow \frac{1}{4a} (\partial_\alpha h)^2 &= \frac{1}{a} f_2^2 \cdot \alpha^2 + \frac{1}{4a} f_1^2 + \frac{1}{4a} u^2 \cdot \beta^2 + \frac{1}{a} f_1 f_2 \cdot \alpha + \frac{1}{a} f_2 u \cdot \alpha\beta + \frac{1}{2a} f_1 u \cdot \beta \\ \frac{1}{4b} (\partial_\beta h)^2 &= \frac{1}{b} g_2^2 \cdot \beta^2 + \frac{1}{4b} g_1^2 + \frac{1}{4b} u^2 \cdot \alpha^2 + \frac{1}{b} g_1 g_2 \cdot \beta + \frac{1}{b} g_2 u \cdot \alpha\beta + \frac{1}{2b} g_1 u \cdot \alpha \end{aligned}$$

Plugging these into the HJB equation yields:

$$\begin{aligned}
& \underbrace{\left\{ (\partial_t + \mathcal{L}^\varepsilon) f_2 - (\phi + \psi) + \frac{1}{a} f_2^2 + \frac{1}{4b} u^2 \right\}}_{(1)} \alpha^2 + \underbrace{\left\{ (\partial_t + \mathcal{L}^\varepsilon) f_1 - \kappa \varepsilon + \frac{1}{a} f_1 f_2 + \frac{1}{2b} g_1 u \right\}}_{(4)} \alpha \\
& + \underbrace{\left\{ (\partial_t + \mathcal{L}^\varepsilon) g_2 - (\phi + \psi) + \frac{1}{b} g_2^2 + \frac{1}{4a} u^2 \right\}}_{(2)} \beta^2 + \underbrace{\left\{ (\partial_t + \mathcal{L}^\varepsilon) g_1 + \kappa \varepsilon + \frac{1}{b} g_1 g_2 + \frac{1}{2a} f_1 u \right\}}_{(5)} \beta \\
& + \underbrace{\left\{ (\partial_t + \mathcal{L}^\varepsilon) u - 2\psi + \frac{1}{a} f_2 u + \frac{1}{b} g_2 u \right\}}_{(3)} \alpha \beta + \underbrace{\left\{ (\partial_t + \mathcal{L}^\varepsilon) h_0 + \frac{1}{4a} f_1^2 + \frac{1}{4b} g_1^2 \right\}}_{(6)} = 0
\end{aligned}$$

Terminal conditions:  $f_2(T, \varepsilon) = g_2(T, \varepsilon) = -c$   
 $f_1(T, \varepsilon) = g_1(T, \varepsilon) = u(T, \varepsilon) = h_0(T, \varepsilon) = 0$

Since this must hold for every value of  $\alpha$  and  $\beta$ , all terms in braces must vanish individually, which gives a coupled system of PDEs for the unknown functions. We can solve for  $f_2, g_2$  and  $u$  using equations (1)-(3). The three equations have no source terms in  $\varepsilon$  and the corresponding terminal conditions are independent of  $\varepsilon$  as well, implying that the three functions only depend on time. This reduces the problem to solving a coupled system of three ODEs, which we can be done numerically:

$$\begin{cases}
\partial_t f_2 - (\phi + \psi) + \frac{1}{a} f_2^2 + \frac{1}{4b} u^2 = 0 \\
\partial_t g_2 - (\phi + \psi) + \frac{1}{b} g_2^2 + \frac{1}{4a} u^2 = 0 \\
\partial_t u - 2\psi + \frac{1}{a} f_2 u + \frac{1}{b} g_2 u = 0 \\
f_2(T) = g_2(T) = -c; \quad u(T) = 0
\end{cases} \quad (20)$$

Given the solutions to  $f_2, g_2$  and  $u$ , we now can use (4) and (5) to find  $f_1$  and  $g_1$  by solving the following coupled system of PDEs:

$$\begin{cases}
(\partial_t + \mathcal{L}^\varepsilon) f_1 - \kappa \varepsilon + \frac{1}{a} f_1 f_2 + \frac{1}{2b} g_1 u = 0 \\
(\partial_t + \mathcal{L}^\varepsilon) g_1 + \kappa \varepsilon + \frac{1}{b} g_1 g_2 + \frac{1}{2a} f_1 u = 0 \\
f_1(T, \varepsilon) = g_1(T, \varepsilon) = 0
\end{cases}$$

We can simplify these equations by exploiting the affine structure of the model for  $\varepsilon$  and assume that  $f_1$  and  $g_1$  are linear in  $\varepsilon$ :

$$\begin{aligned}
& f_1(t, \varepsilon) = f_{10}(t) + f_{11}(t) \cdot \varepsilon & g_1(t, \varepsilon) = g_{10}(t) + g_{11}(t) \cdot \varepsilon \\
\Rightarrow & (\partial_t + \mathcal{L}^\varepsilon) f_1 = \partial_t f_{10} + \partial_t f_{11} \cdot \varepsilon - \kappa \varepsilon f_{11} & (\partial_t + \mathcal{L}^\varepsilon) g_1 = \partial_t g_{10} + \partial_t g_{11} \cdot \varepsilon - \kappa \varepsilon g_{11}
\end{aligned}$$

Plugging the expressions above into the last PDE system we have:

$$\begin{cases} \partial_t f_{10} + \partial_t f_{11} \cdot \varepsilon - \kappa f_{11} \cdot \varepsilon - \kappa \cdot \varepsilon + \frac{1}{a} f_{10} f_2 + \frac{1}{a} f_{11} f_2 \cdot \varepsilon + \frac{1}{2b} g_{10} u + \frac{1}{2b} g_{11} u \cdot \varepsilon = 0 \\ \partial_t g_{10} + \partial_t g_{11} \cdot \varepsilon - \kappa g_{11} \cdot \varepsilon + \kappa \cdot \varepsilon + \frac{1}{b} g_{10} g_2 + \frac{1}{b} g_{11} g_2 \cdot \varepsilon + \frac{1}{2a} f_{10} u + \frac{1}{2a} f_{11} u \cdot \varepsilon = 0 \\ f_{10}(T) = f_{11}(T) = g_{10}(T) = g_{11}(T) = 0 \end{cases}$$

Collecting terms we have:

$$\begin{cases} \left\{ \partial_t f_{10} + \frac{1}{a} f_{10} f_2 + \frac{1}{2b} g_{10} u \right\} + \left\{ \partial_t f_{11} - \kappa f_{11} - \kappa + \frac{1}{a} f_{11} f_2 + \frac{1}{2b} g_{11} u \right\} \cdot \varepsilon = 0 \\ \left\{ \partial_t g_{10} + \frac{1}{b} g_{10} g_2 + \frac{1}{2a} f_{10} u \right\} + \left\{ \partial_t g_{11} - \kappa g_{11} + \kappa + \frac{1}{b} g_{11} g_2 + \frac{1}{2a} f_{11} u \right\} \cdot \varepsilon = 0 \\ f_{10}(T) = f_{11}(T) = g_{10}(T) = g_{11}(T) = 0 \end{cases}$$

Once again, all the terms in the braces must vanish individually, which yields two coupled systems of ODEs. From the ‘‘constant’’ terms we can solve for  $f_{10}$  and  $g_{10}$ :

$$\begin{cases} \partial_t f_{10} + \left(\frac{f_2}{a}\right) f_{10} + \left(\frac{u}{2b}\right) g_{10} = 0 \\ \partial_t g_{10} + \left(\frac{g_2}{b}\right) g_{10} + \left(\frac{u}{2a}\right) f_{10} = 0 \\ f_{10}(T) = g_{10}(T) = 0 \end{cases}$$

for which we have the trivial solution  $f_{10}(t) = 0$  and  $g_{10}(t) = 0$ . Focusing on the two remaining  $\varepsilon$  terms, we are left with the following ODE system, which can also be solved numerically:

$$\begin{cases} \partial_t f_{11} + \left(\frac{f_2}{a} - \kappa\right) f_{11} + \left(\frac{u}{2b}\right) g_{11} - \kappa = 0 \\ \partial_t g_{11} + \left(\frac{g_2}{b} - \kappa\right) g_{11} + \left(\frac{u}{2a}\right) f_{11} + \kappa = 0 \\ f_{11}(T) = g_{11}(T) = 0 \end{cases} \quad (21)$$

Once we solve for the functions in the ODE systems (20) and (21), we can write the optimal controls by substituting the last ansatz into the feedback form solution given in (18) and (19):

$$\nu_t^* = \frac{f_2}{a} \cdot \alpha_t + \frac{f_{11}}{2a} \cdot \varepsilon_t + \frac{u}{2a} \cdot \beta_t \quad (22)$$

$$\mu_t^* = \frac{g_2}{b} \cdot \beta_t + \frac{g_{11}}{2b} \cdot \varepsilon_t + \frac{u}{2b} \cdot \alpha_t \quad (23)$$

#### 4.2. Case 2: $\phi = \mathbf{0}$ , $\psi > \mathbf{0}$

If we assume that the agent does not have a relative inventory running penalty, i.e.  $\psi = 0$ , we can derive a closed-form solution for the optimal trading rate controls. Under this assumption, we would follow the same steps as in the previous case while leaving out the second running penalty term. The result would be equivalent to setting  $\psi = 0$  and excluding the function  $u$  associated with the cross-term  $\alpha\beta$ . Thus, the optimal controls in this case



require solving modified versions of the ODE systems (20) and (21). First, we must solve the ODE system:

$$\begin{cases} (\partial_t + \mathcal{L}^\varepsilon)f_2 - \phi + \frac{1}{a}f_2^2 = 0 \\ (\partial_t + \mathcal{L}^\varepsilon)g_2 - \phi + \frac{1}{b}g_2^2 = 0 \\ f_2(T) = g_2(T) = -c \end{cases}$$

Notice that the form is similar to that of the Almgren-Chriss optimal execution problem. We adapt the solutions to that problem to obtain expressions for  $f_2$  and  $g_2$  as follows:

$$f_2(t) = \sqrt{a\phi} \cdot \frac{1 + \zeta_1 \cdot e^{2\gamma_1(T-t)}}{1 - \zeta_1 \cdot e^{2\gamma_1(T-t)}}; \quad \gamma_1 = \sqrt{\phi/a}, \quad \zeta_1 = \frac{c + \sqrt{a\phi}}{c - \sqrt{a\phi}} \quad (24)$$

$$g_2(t) = \sqrt{b\phi} \cdot \frac{1 + \zeta_2 \cdot e^{2\gamma_2(T-t)}}{1 - \zeta_2 \cdot e^{2\gamma_2(T-t)}}; \quad \gamma_2 = \sqrt{\phi/b}, \quad \zeta_2 = \frac{c + \sqrt{b\phi}}{c - \sqrt{b\phi}} \quad (25)$$

Given  $f_2$  and  $g_2$  we must then solve a second ODE system given by:

$$\begin{cases} (\partial_t + \mathcal{L}^\varepsilon)f_1 - \kappa\varepsilon + \frac{f_2}{a}f_1 = 0 \\ (\partial_t + \mathcal{L}^\varepsilon)g_1 + \kappa\varepsilon + \frac{g_2}{b}g_1 = 0 \\ f_1(T, \varepsilon) = g_1(T, \varepsilon) = 0 \end{cases}$$

Once again we can exploit the fact that this ODE system appears in the context of the optimal execution problem with order flow. Thus, we can adapt the solutions to that problem to obtain expressions for  $f_1$  and  $g_1$ :

$$f_1(t, \varepsilon) = -\kappa \cdot \omega(T - t, \zeta_1, \gamma_1) \cdot \varepsilon \quad (26)$$

$$g_1(t, \varepsilon) = \kappa \cdot \omega(T - t, \zeta_2, \gamma_2) \cdot \varepsilon \quad (27)$$

where

$$\omega(\tau, \zeta, \gamma) = \frac{1}{\zeta} \frac{1}{e^{\gamma\tau} - e^{-\gamma\tau}} \left[ \zeta e^{\gamma\tau} \frac{1 - e^{-(\kappa+\gamma)\tau}}{\kappa + \gamma} - e^{-\gamma\tau} \frac{1 - e^{-(\kappa-\gamma)\tau}}{\kappa - \gamma} \right] \geq 0$$

Finally, we can write the optimal controls in this case by substituting these functions into the feedback form solution given in (18) and (19):

$$\nu_t^* = \frac{f_2}{a} \cdot \alpha_t - \frac{\kappa}{2a} \cdot \omega(T - t, \zeta_1, \gamma_1) \cdot \varepsilon_t \quad (28)$$

$$\mu_t^* = \frac{g_2}{b} \cdot \beta_t + \frac{\kappa}{2b} \cdot \omega(T - t, \zeta_2, \gamma_2) \cdot \varepsilon_t \quad (29)$$

### 4.3. Case 3: $\phi = \mathbf{0}$ , $\psi = \mathbf{0}$

Another case in which we can derive a closed-form solution is one where we exclude both absolute and relative inventory penalties by setting  $\phi = 0$  and  $\psi = 0$ . Similar to the previous case, the solution in this case involves solving modified versions of the ODE systems (20) and (21). First, we must solve:

$$\begin{cases} (\partial_t + \mathcal{L}^\varepsilon)f_2 + \frac{1}{a}f_2^2 = 0 \\ (\partial_t + \mathcal{L}^\varepsilon)g_2 + \frac{1}{b}g_2^2 = 0 \\ f_2(T) = g_2(T) = -c \end{cases}$$

This system arises in the optimal execution problem with temporary price impact only. The solution is obtained by direct integration and is given as follows:

$$f_2(t) = - \left[ \frac{1}{c} + \frac{1}{a}(T-t) \right]^{-1} \quad (30)$$

$$g_2(t) = - \left[ \frac{1}{c} + \frac{1}{b}(T-t) \right]^{-1} \quad (31)$$

Given  $f_2$  and  $g_2$  we must then solve a second ODE system given by:

$$\begin{cases} (\partial_t + \mathcal{L}^\varepsilon)f_1 - \kappa\varepsilon + \frac{f_2}{a}f_1 = 0 \\ (\partial_t + \mathcal{L}^\varepsilon)g_1 + \kappa\varepsilon + \frac{g_2}{b}g_1 = 0 \\ f_1(T, \varepsilon) = g_1(T, \varepsilon) = 0 \end{cases}$$

for which the solution is given by  $f_1(t, \varepsilon) = f_{11}(t) \cdot \varepsilon$  and  $g_1(t, \varepsilon) = g_{11}(t) \cdot \varepsilon$  where  $f_{11}$  and  $g_{11}$  satisfy:

$$\begin{cases} \partial_t f_{11} + \left( \frac{f_2}{a} - \kappa \right) f_{11} - \kappa = 0 \\ \partial_t g_{11} + \left( \frac{g_2}{b} - \kappa \right) g_{11} + \kappa = 0 \\ f_{11}(T) = g_{11}(T) = 0 \end{cases}$$

The solution to this system (details are deferred to the appendix) is given by:

$$f_{11}(t) = -1 + \left[ \frac{1}{c} + \frac{1}{a}(T-t) \right]^{-1} \cdot \left[ \frac{1}{c} e^{-\kappa(T-t)} + \frac{1}{\kappa a} (1 - e^{-\kappa(T-t)}) \right] \quad (32)$$

$$g_{11}(t) = 1 - \left[ \frac{1}{c} + \frac{1}{b}(T-t) \right]^{-1} \cdot \left[ \frac{1}{c} e^{-\kappa(T-t)} + \frac{1}{\kappa b} (1 - e^{-\kappa(T-t)}) \right] \quad (33)$$

And as usual we can substitute these terms into the feedback form solution given by (18) and (19) to obtain the optimal trading rates for this case:

$$\nu_t^* = -\frac{1}{a} \left[ \frac{1}{c} + \frac{1}{a}(T-t) \right]^{-1} \cdot \alpha_t - \frac{1}{2a} \left( 1 - \left[ \frac{1}{c} + \frac{1}{a}(T-t) \right]^{-1} \cdot \left[ \frac{1}{c} e^{-\kappa(T-t)} + \frac{1}{\kappa a} (1 - e^{-\kappa(T-t)}) \right] \right) \cdot \varepsilon_t \quad (34)$$

$$\mu_t^* = -\frac{1}{b} \left[ \frac{1}{c} + \frac{1}{b}(T-t) \right]^{-1} \cdot \beta_t + \frac{1}{2b} \left( 1 - \left[ \frac{1}{c} + \frac{1}{b}(T-t) \right]^{-1} \cdot \left[ \frac{1}{c} e^{-\kappa(T-t)} + \frac{1}{\kappa b} (1 - e^{-\kappa(T-t)}) \right] \right) \cdot \varepsilon_t \quad (35)$$

## 5. Discussion

We can use the closed-form solutions for Cases 2 and 3 obtained in the previous section to begin our discussion of the optimal trading rates, and then consider the effect of including both running penalties using numerical solutions to the ODE systems (20) and (21).

### 5.1. General Structure of Optimal Trading Rates

In all three cases, the optimal trading rates for the two assets have a similar structure: for each asset there is a base trading rate associated with the level of mispricing (the prevailing value of the cointegration factor), and an inventory adjustment term which adjusts the trading rate depending on the current level of holdings associated with that asset. In Case 1, there is an additional adjustment term that depends on the level of holdings of the other asset, i.e. the trading rate for Asset 1 also depends on the inventory level of Asset 2 and vice versa. These terms are summarized for the three cases in the tables below:

This structure has an intuitive interpretation: in Cases 2 and 3 the base trading terms for the two assets always have opposite signs,<sup>1</sup> meaning that the agent always trades the two assets in opposite directions. It is also intuitive that the trading rate is affected by inventory, since this will account for transaction and liquidation costs. In particular, the inventory adjustment terms for Cases 2 and 3 are always negative,<sup>2</sup> meaning that the agent will slow down her trading in an asset when she has a long position and speed up when she has a short position. This is in line with the intuition for ensuring that the inventories do not grow so large as to increase transaction and liquidation costs. In Case 2, there is also the added effect of ensuring that absolute holdings do not become too large and result in large running penalties. Finally, the cross-dependence of trading rates on inventories of the other asset in Case 1 is driven by the relative inventory penalty; since the agent is trying to keep the holdings equal in magnitude and opposite in direction, the trading rate of one asset must be informed by the holdings in the other. Figure 1 demonstrates the properties of the HJB solutions discussed above.

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<sup>1</sup>This can be deduced by noticing that in Case 2 we have  $\omega > 0$ , and for Case 3 the expression in the outermost brackets is positive; this can be seen through term-by-term comparison of the expressions in the two square brackets, concluding that their product must in fact be less than 1.

<sup>2</sup>This is clear to see for Case 3. For Case 2, it follows from the fact that  $\gamma_i > 0$  and  $|\zeta_i| > 1$  which implies that  $f_2, g_2 < 0$ . Various numerical experiments show that is the case for Case 1 also.

<b>Case 1</b>	$\nu_1^*$	$\mu_1^*$
Base Trading Term	$\frac{f_{11}}{2a}$	$\frac{g_{11}}{2b}$
$\alpha$ Adjustment	$\frac{f_2}{a}$	$\frac{u}{2b}$
$\beta$ Adjustment	$\frac{u}{2a}$	$\frac{g_2}{b}$

where  $f_2, g_2, u$  solve (20) and  $f_{11}, g_{11}$  solve (21).

<b>Case 2</b>	$\nu_2^*$	$\mu_2^*$
Base Trading Term	$-\frac{\kappa}{2a} \cdot \omega(T-t, \zeta_1, \gamma_1)$	$\frac{\kappa}{2b} \cdot \omega(T-t, \zeta_2, \gamma_2)$
$\alpha$ Adjustment	$\frac{1+\zeta_1 \cdot e^{2\gamma_1(T-t)}}{1-\zeta_1 \cdot e^{2\gamma_1(T-t)}}$	-
$\beta$ Adjustment	-	$\frac{1+\zeta_2 \cdot e^{2\gamma_2(T-t)}}{1-\zeta_2 \cdot e^{2\gamma_2(T-t)}}$

<b>Case 3</b>	$\nu_3^*$
Base Trading Term	$-\frac{1}{2a} \left( 1 - \frac{1}{a} \left[ \frac{1}{c} + \frac{1}{a}(T-t) \right]^{-1} \cdot \left[ \frac{1}{c} e^{-\kappa(T-t)} + \frac{1}{\kappa a} (1 - e^{-\kappa(T-t)}) \right] \right)$
$\alpha$ Adjustment	$-\frac{1}{a} \left[ \frac{1}{c} + \frac{1}{a}(T-t) \right]^{-1}$
$\beta$ Adjustment	-
	$\mu_3^*$
Base Trading Term	$\frac{1}{2b} \left( 1 - \frac{1}{b} \left[ \frac{1}{c} + \frac{1}{b}(T-t) \right]^{-1} \cdot \left[ \frac{1}{c} e^{-\kappa(T-t)} + \frac{1}{\kappa b} (1 - e^{-\kappa(T-t)}) \right] \right)$
$\alpha$ Adjustment	-
$\beta$ Adjustment	$-\frac{1}{b} \left[ \frac{1}{c} + \frac{1}{b}(T-t) \right]^{-1}$

Table 1: Summary of terms in the optimal trading rates (the subscripts associated with  $\nu^*$  and  $\mu^*$  correspond to the case number)

## 5.2. Comparative Statics

In this section, we consider the effect of varying various model parameters on the optimal trading rates as well as the profit-and-loss (PnL) distribution. To achieve this we start with

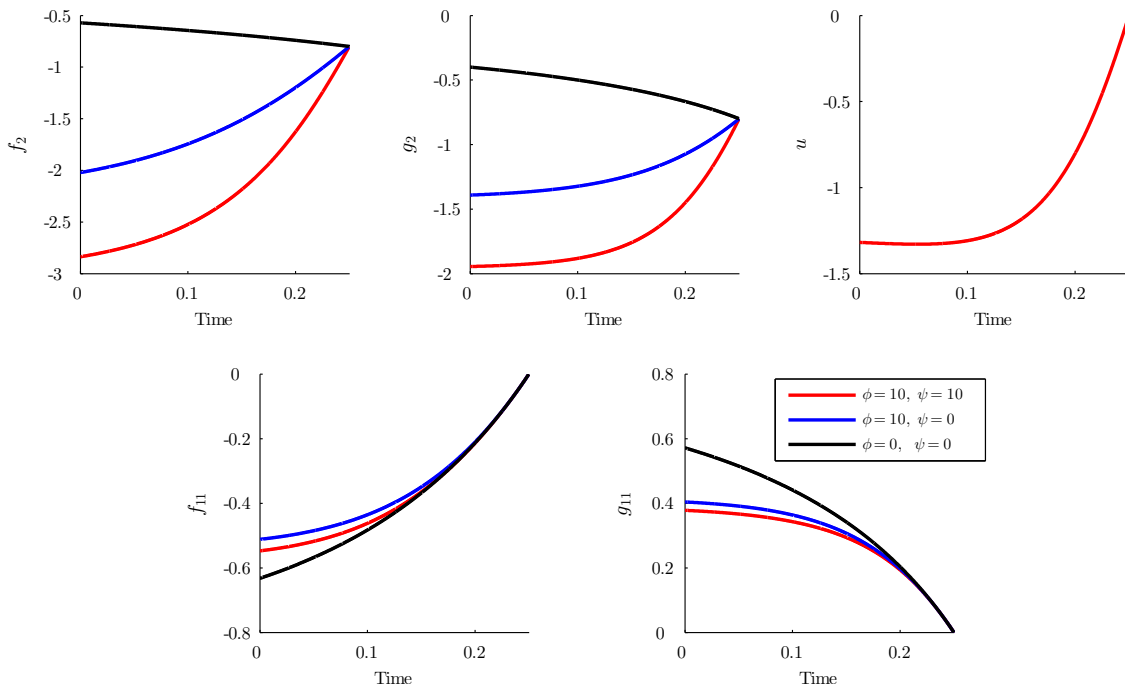


Figure 1: Solutions to HJB equation (top row from left:  $f_2$ ,  $g_2$ ,  $u$ ; bottom row from left:  $f_{11}$ ,  $g_{11}$ ) assuming  $T = 0.25$ ,  $a = 0.5$ ,  $b = 0.2$ ,  $c = 0.1$ ,  $\kappa = 5$ ,  $\sigma = 0.08$ ,  $\eta = 0.11$ ,  $\rho = -0.3$ . Case 1 corresponds to the red line with  $\phi = 10, \psi = 10$  and is computed numerically. Cases 2 and 3 assume  $\phi = 10, \psi = 0$  and  $\phi = 0, \psi = 0$ , respectively, correspond to the blue and black lines and are computed according to the solutions given in sections 4.2 and 4.3.

the following set of parameters (the base case):

**Asset/Factor Dynamics:**  $\kappa = 5$ ,  $\sigma = 0.08$ ,  $\eta = 0.11$ ,  $\rho = -0.3$ ,  
 $S_0^1 = 25$ ,  $S_0^2 = 20$

**Trading Parameters:**  $a = 0.1$ ,  $b = 0.1$ ,  $c = 0.1$ ,  $T = 0.25$

**Simulation Parameters:** Number of simulations = 1000, Number of steps = 1000

In this setup, the agent has an investment horizon of  $T = 0.25$  (3 months), and can trade in two assets priced at 25 and 20, with annualized volatilities of 23% and 30%, respectively, and an instantaneous correlation of approximately 0.3. As mentioned in the outset, given the form of the cointegration factor and the fact that it has a mean-reversion level of zero, both assets will converge to a price of 22.5 in the long run. Clearly, Asset 1 is overpriced and Asset 2 is underpriced. The mean reversion rate is given by  $\kappa$  which, in this case, corresponds to a half-life<sup>3</sup> of approximately 1.5 months. These properties can be seen in the simulated paths shown in Figure 2.

<sup>3</sup>The half-life of the mean reversion is the average amount of time it will take for the process to get pulled half-way back to its mean level. This is given by  $\frac{\log 2}{\kappa}$ .

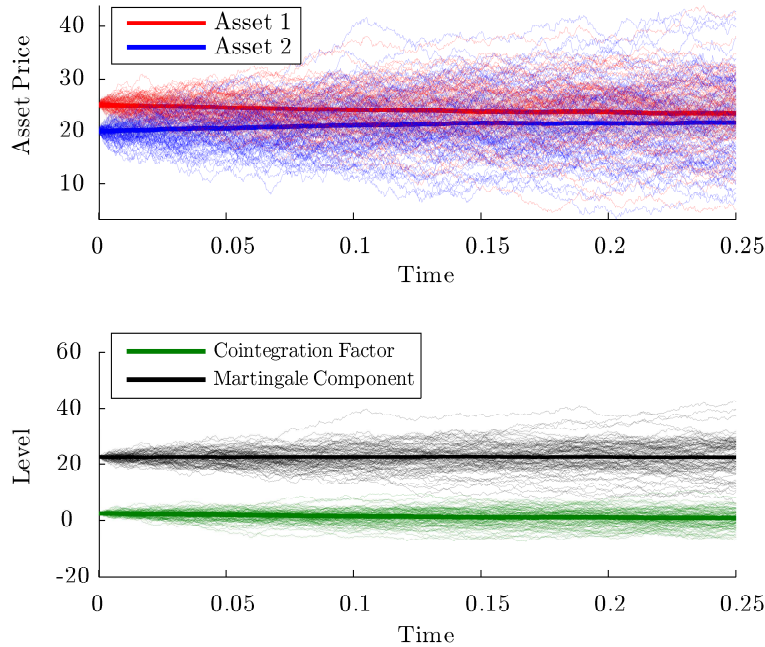


Figure 2: Simulated asset prices (top panel) and factor dynamics (bottom panel). Each transparent line represents a single simulation and the bold lines correspond to the average price/level across simulations (only 100 simulations are shown).

Once we simulate the underlying processes, we can compute the trading rules (which are random due to their dependence on the cointegration factor), and track the evolution of the cash and asset holdings through time in each simulation. At the terminal date, we compute the total PnL in each simulation according to the terminal reward given in the performance criteria (15). These quantities are represented in Figures 3 and 4.

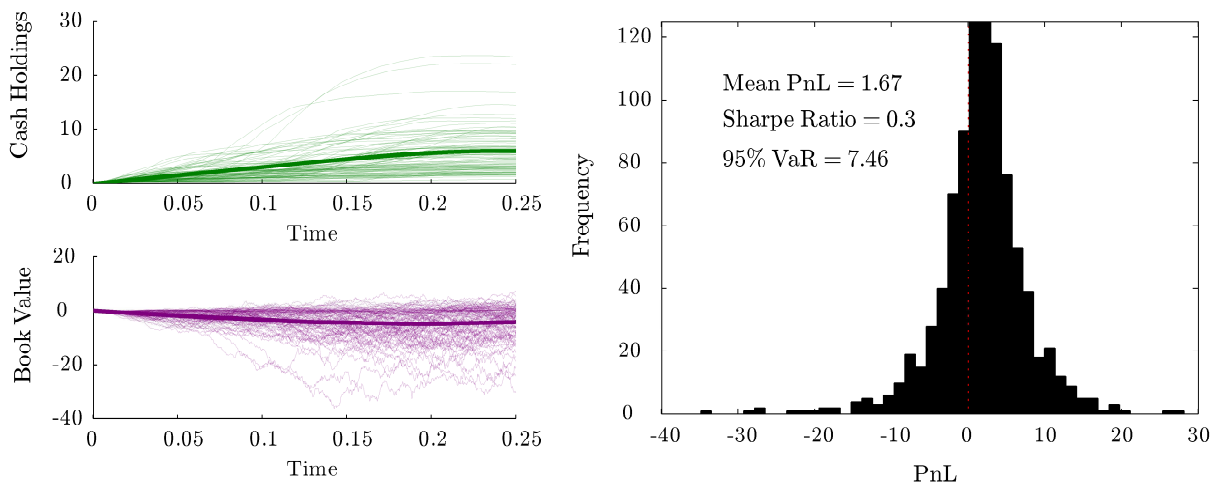


Figure 4: Cash holdings and book value of asset holdings through time (left panel; only 100 simulations are shown); PnL based on 1000 simulations (right panel); Sharpe Ratio is the ratio of the average PnL to the standard deviation of the PnL across simulations

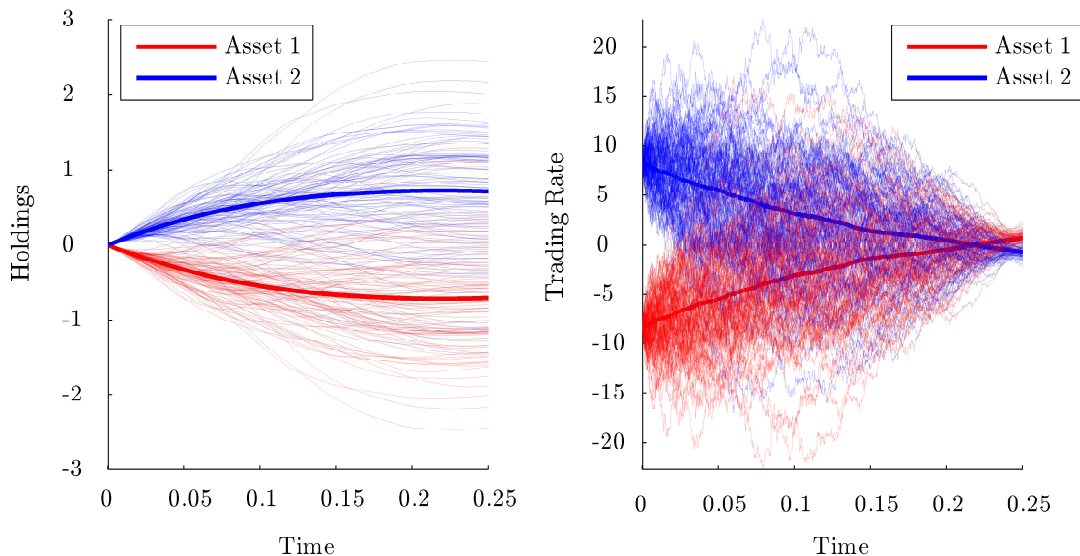
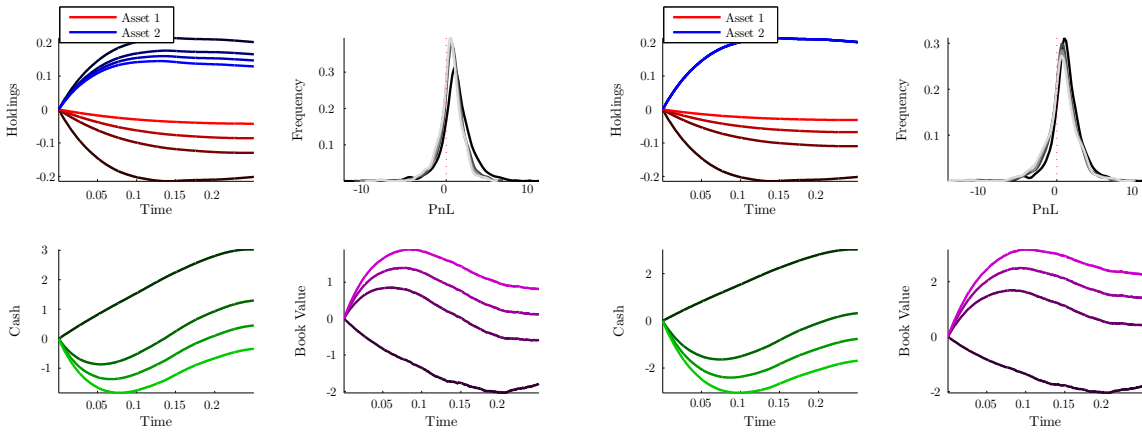


Figure 3: Optimal holdings (left panel) and trading rates (right panel) based on simulated asset prices. Each transparent line represents a single simulation and the bold lines correspond to the average holdings/trading rate across simulations (only 100 simulations are shown).

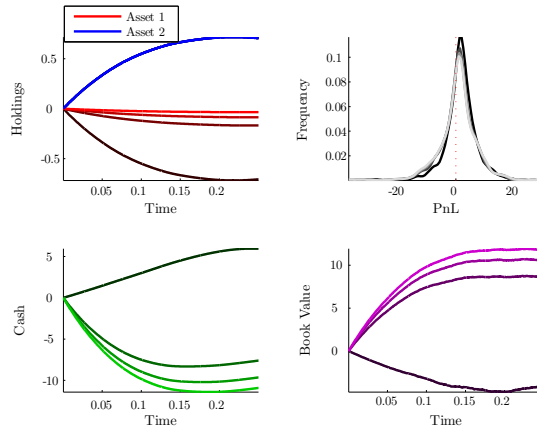
### 5.2.1. Transaction Costs

When temporary price impact is equal for the two assets then, in all three cases, the base trading rates are equal in magnitude and opposite in direction and the inventory adjustments are equal. This can be clearly seen for Cases 2 and 3 by examining Table 1, and it can be deduced for Case 1 by noticing the symmetry in the ODE systems (20) and (21) when  $a = b$  and through inspection of the numerical solution. This is also demonstrated by the trading pattern for the example shown in Figure 3. This trading “symmetry” is also driven by the form of the cointegration factor. If the integration factor was of a more general form, i.e.  $\varepsilon_t = AS_t^1 + BS_t^2$ , then we would expect this symmetry to be broken even when transaction costs are equal. Figure 5 shows the effect of increasing the transaction cost associated with Asset 1 assuming the parameters in the base case and three different pairs of inventory aversion parameters corresponding to each of the three cases discussed in the previous section.



(a) Case 1:  $\phi = 10, \psi = 10$

(b) Case 2:  $\phi = 10, \psi = 0$



(c) Case 3:  $\phi = 0, \psi = 0$

Figure 5: Effect of increasing the temporary impact cost of Asset 1 on (clockwise from top left) holdings, PnL distribution, book value, and cash holdings. Three cases are shown corresponding to different  $(\phi, \psi)$  pairs; values of  $a$  considered are: 0.1, 0.5, 1, 2.5 (lighter lines correspond to increasing values of  $a$ ).

The plots show that there is a decrease in the level of trading associated with Asset 1, which is the expected result since trading in that asset has become more expensive. In Case 2 and 3 the trading rate for Asset 2 is unaffected. However, in Case 1 there is a slight decrease in the trading rate for Asset 2 due to the presence of the cross-inventory adjustment term. In all cases there is a drop in cash holdings and an increase in the book value of holdings because the agent is decreasing trading activity (short-selling) of Asset 1. It is also interesting to note that this adjustment is more gradual in Cases 1 and 2 than Case 3, suggesting that the absence of inventory penalties makes the investor more sensitive to transaction costs. The effect on the PnL distribution is minimal.

We can also consider the effect of increasing the liquidation cost,  $c$ , as summarized in Figure 6. As the value of  $c$  increases the trading activity in both assets falls, with the



decrease in trading activity being greater for Cases 1 and 2 due to the presence of running inventory penalties. Also, since  $a = b$  in the base case, changing the value of  $c$  does not break the trading symmetry discussed earlier. Note that this induced symmetry also leads to the optimal controls in Cases 1 and 2 being identical, since the additional relative inventory penalty remains at zero throughout the investment horizon. Additionally, decreasing cash holdings and book values (which are lower in Cases 1 and 2 than Case 3) are a consequence of reduced trading activity.

With regards to the PnL distribution, we find that the PnL Sharpe ratio stays almost constant as the mean and standard deviation fall with increased liquidation cost. Interestingly, the distribution becomes more negatively skewed with lower levels of 95% Value-at-Risk. The increase in liquidation costs reduces trading activity which indirectly makes the agent more cautious.

Sharpe Ratio					95% Value-at-Risk				
$c$	0.1	0.5	1.01	2.5	$c$	0.1	0.5	1.01	2.5
<b>Case 1 &amp; 2</b>	0.58	0.57	0.56	0.54	<b>Case 1 &amp; 2</b>	1.92	1.77	1.64	1.47
<b>Case 3</b>	0.30	0.32	0.33	0.35	<b>Case 3</b>	7.46	5.01	3.97	3.11

Table 2: Effect of increasing liquidation cost,  $c$ , on PnL Sharpe ratio (left panel) and 95% Value-at-Risk (right panel). The values of the aversion parameters are  $\phi = 10$ ,  $\psi = 10$  for Case 1,  $\phi = 10$ ,  $\psi = 0$  for Case 2 and  $\phi = 0$ ,  $\psi = 0$  for Case 3. Note that the results for Cases 1 and 2 are identical.

As an aside, for any given liquidation cost amount, the inventory penalties in Cases 1 and 2 lead to a significant reduction in 95% VaR compared to Case 3. This is because the agent is able to reduce the book value of her holdings, and the smaller inventory levels translate to a lower level of absolute risk.

### 5.2.2. Aversion Parameters

Now we consider the impact of varying the aversion parameters associated with absolute and relative inventory penalties. Before examining this, however, we adjust the base case slightly by changing the temporary impact parameter for Asset 2 to  $b = 0.25$  in order to break the symmetry in the trading patterns. Also, we consider the effect of increasing  $\phi$  when  $\psi$  is zero compared to when  $\psi$  is non-zero, and vice versa. Figures 7 and 8 along with Tables 3 and 4 show the results for these cases.

As  $\phi$  increases we find that the magnitude of asset and cash holdings are reduced along with the total book value through time, which is the expected result when increasing absolute

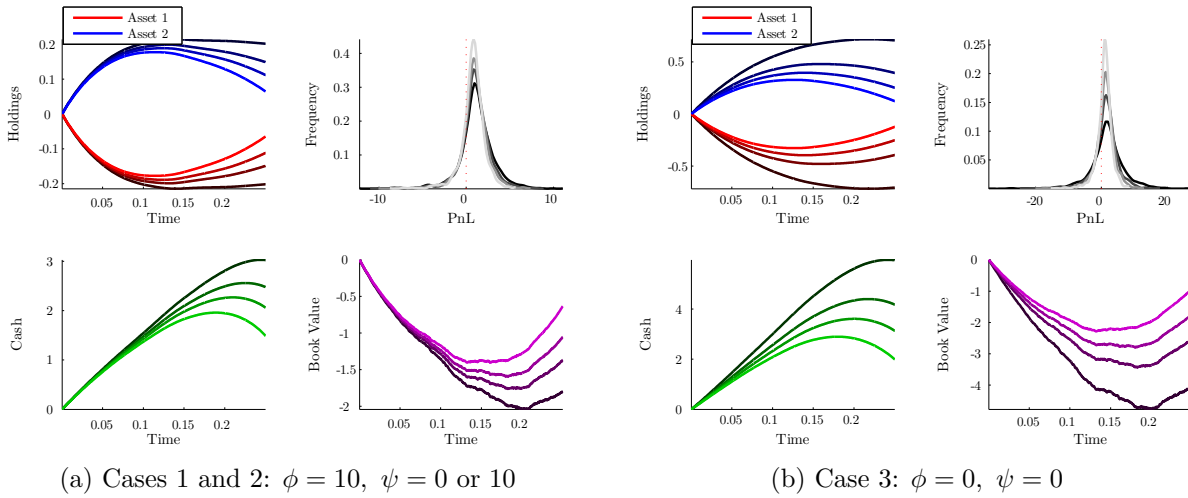


Figure 6: Effect of increasing the liquidation cost on (clockwise from top left) holdings, PnL distribution, book value, and cash holdings. Three cases are shown corresponding to different  $(\phi, \psi)$  pairs; Cases 1 and 2 result in identical results (left panel). Values of  $c$  considered are: 0.1, 0.5, 1.01, 2.5 (lighter lines correspond to increasing values of  $c$ ).

inventory aversion. This also occurs when  $\psi > 0$  but there is added symmetry in the trading pattern which stems from the relative inventory penalty. This penalty also leads to a further decrease in the magnitude of holdings. Additionally, the range of the PnL distribution decreases dramatically as the level of absolute risk is reduced. Table 3 shows a substantial increase in the Sharpe ratio accompanied by a reduction in the 95% Value-at-Risk.

When  $\psi$  is increased, the trading pattern in the two assets becomes more symmetric, as expected. This can be seen in Figure 8 as the holdings in both assets shift upwards. We find that the magnitude of cash holdings as well as the total book value through time are reduced. This also occurs when  $\phi > 0$  except with the magnitudes of holdings reduced even further due to the absolute inventory penalty. Finally, we find by examining Table 4 that increasing  $\psi$  has limited impact on the PnL distribution, which appears to be driven mostly by the absolute aversion parameter  $\phi$ .

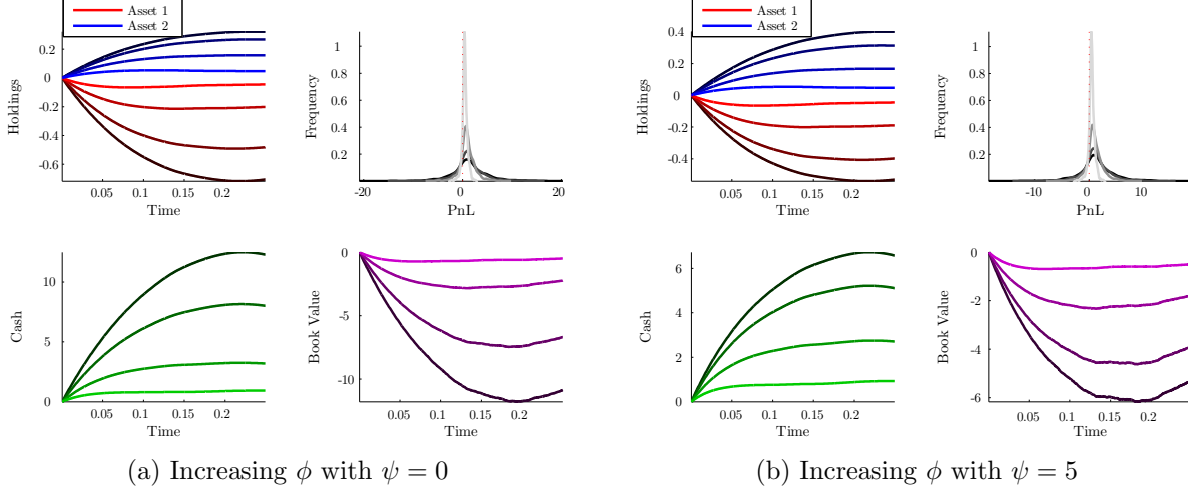


Figure 7: Effect of increasing the absolute inventory aversion parameter on (clockwise from top left) holdings, PnL distribution, book value, and cash holdings. Two cases are shown corresponding to  $\psi = 0$  (left panel) and  $\psi = 5$  (right panel); values of  $\phi$  considered are: 0, 2, 10, 50 (lighter lines correspond to increasing values of  $\phi$ ).

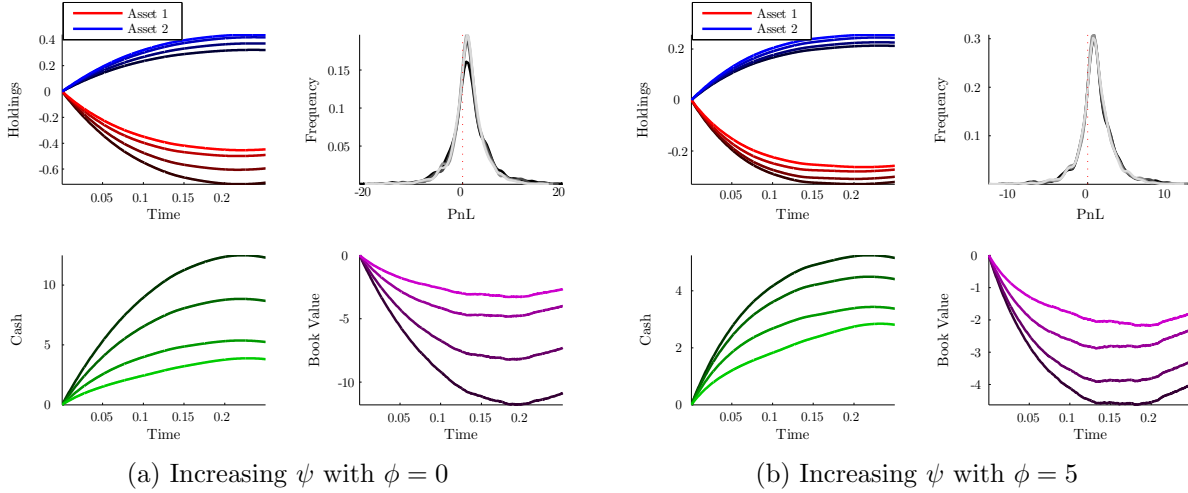


Figure 8: Effect of increasing the relative inventory aversion parameter on (clockwise from top left) holdings, PnL distribution, book value, and cash holdings. Two cases are shown corresponding to  $\phi = 0$  (left panel) and  $\phi = 5$  (right panel); values of  $\psi$  considered are: 0, 2, 10, 50 (lighter lines correspond to increasing values of  $\psi$ ).

Sharpe Ratio					95% Value-at-Risk				
$\phi$	0	2	10	50	$\phi$	0	2	10	50
$\psi = 0$	0.31	0.40	0.57	0.78	$\psi = 0$	5.24	3.47	1.57	0.41
$\psi = 5$	0.34	0.41	0.56	0.78	$\psi = 5$	4.30	3.22	1.56	0.42

Table 3: Effect of increasing absolute inventory aversion parameter,  $\phi$ , on PnL Sharpe ratio (left panel) and 95% Value-at-Risk (right panel).

Sharpe Ratio					95% Value-at-Risk				
$\psi$	0	2	10	50	$\psi$	0	2	10	50
$\phi = 0$	0.31	0.34	0.34	0.31	$\phi = 0$	5.24	4.66	4.16	4.50
$\phi = 5$	0.49	0.49	0.48	0.45	$\phi = 5$	2.40	2.41	2.30	2.40

Table 4: Effect of increasing relative inventory aversion parameter,  $\psi$ , on PnL Sharpe ratio (left panel) and 95% Value-at-Risk (right panel).

### 5.2.3. Asset Dynamics

In this section we examine the sensitivity of the optimal solutions to changes in the asset dynamics; namely, the volatilities and correlation of the two assets, and the mean reversion rate. Surprisingly, the optimal solutions in all cases do not depend on the volatility and correlation of the two assets since they do not involve the parameters  $\sigma, \eta$  and  $\rho$ . However, this makes sense when we consider that the agent sets her positions according to the size of the mispricing at any given point in time (distance from the mean level of the cointegration factor), as well as how fast she expects this gap to close (the mean reversion rate).

With respect to the mean reversion rate, we find that a shorter half-life forces the agent to take larger positions and maintain them throughout the investment horizon as shown in Figure 9. This is somewhat tempered in the presence of inventory penalties. The increased trading activity takes place in an effort to take advantage of the mispricing before the spread closes. The sooner the spread is expected to close, the more risk the agent needs to take on in order to profit. Table 5 shows that this increase in risk can be substantial, particularly when there are no risk controls in the form of inventory penalties. This added risk also leads to the deterioration of the PnL Sharpe ratio.

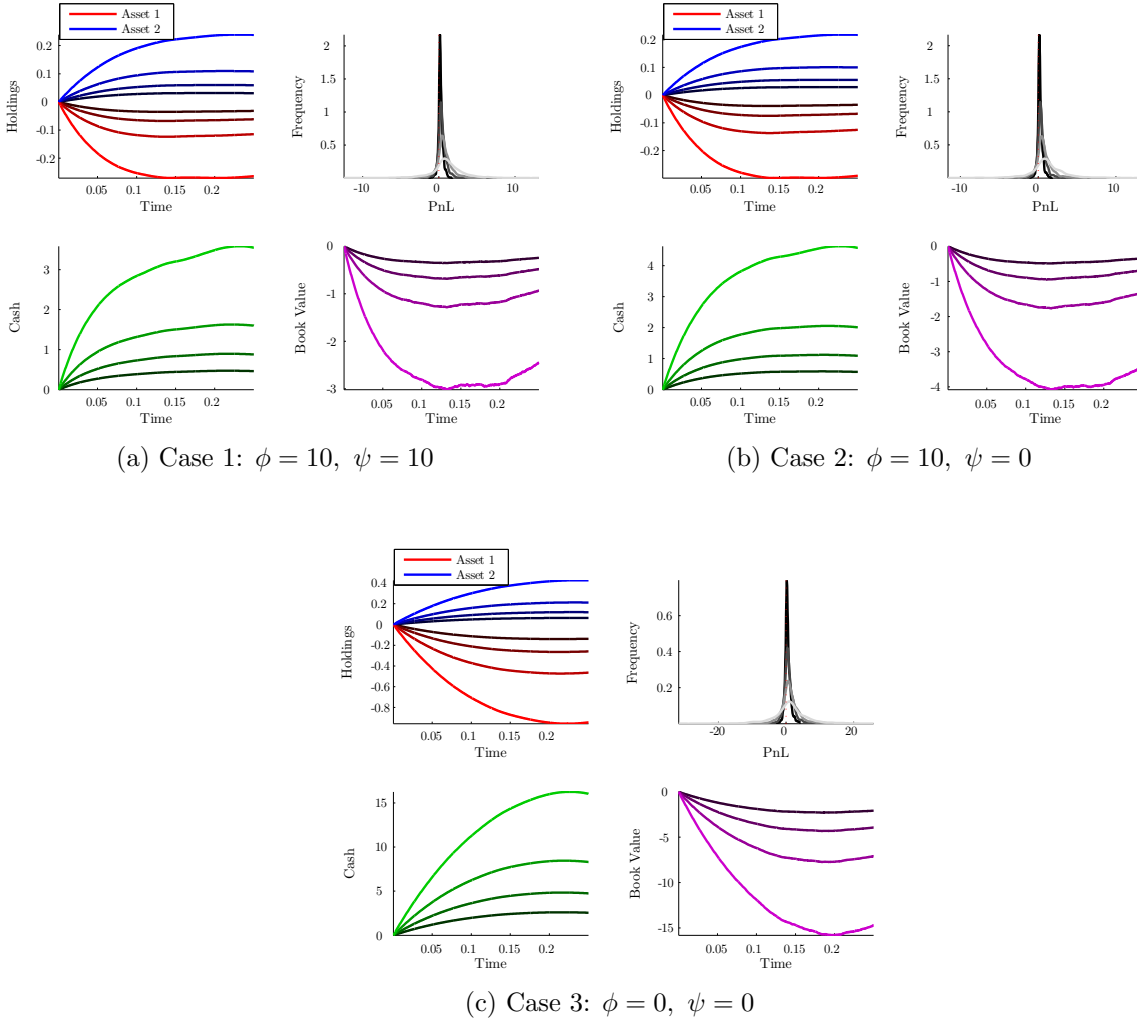


Figure 9: Effect of increasing the mean reversion rate of the cointegration factor on (clockwise from top left) holdings, PnL distribution, book value, and cash holdings. Three cases are shown corresponding to different  $(\phi, \psi)$  pairs; values of  $\kappa$  considered are: 0.7, 1.4, 2.8, 8.3 which correspond to a half-life of 12, 6, 3, and 1 months, respectively (lighter lines correspond to increasing values of  $\kappa$ ).

Sharpe Ratio					95% Value-at-Risk				
$\kappa$	0.7	1.4	2.8	8.3	$\kappa$	0.1	0.5	1.01	2.5
<b>Case 1</b>	0.73	0.70	0.63	0.46	<b>Case 1</b>	0.22	0.44	0.88	2.52
<b>Case 2</b>	0.74	0.71	0.65	0.47	<b>Case 2</b>	0.22	0.44	0.88	2.48
<b>Case 3</b>	0.56	0.51	0.43	0.20	<b>Case 3</b>	0.63	1.27	2.69	8.20

Table 5: Effect of increasing mean reversion rate,  $\kappa$ , on PnL Sharpe ratio (left panel) and 95% Value-at-Risk (right panel). The values of the aversion parameters are  $\phi = 10, \psi = 10$  for Case 1,  $\phi = 10, \psi = 0$  for Case 2 and  $\phi = 0, \psi = 0$  for Case 3.

## Appendix

### *Solution to Case 3*

In this appendix we present the solution to the optimal control problem when there are no inventory penalties (Case 3) in more detail. We are interested in solving the ODE system:

$$\begin{cases} (\partial_t + \mathcal{L}^\varepsilon)f_1 - \kappa\varepsilon + \frac{f_2}{a}f_1 = 0 \\ (\partial_t + \mathcal{L}^\varepsilon)g_1 + \kappa\varepsilon + \frac{g_2}{b}g_1 = 0 \\ f_1(T, \varepsilon) = g_1(T, \varepsilon) = 0 \end{cases}$$

where  $f_2$  and  $g_2$  are given by:

$$\begin{aligned} f_2(t) &= - \left[ \frac{1}{c} + \frac{1}{a}(T-t) \right]^{-1} \\ g_2(t) &= - \left[ \frac{1}{c} + \frac{1}{b}(T-t) \right]^{-1} \end{aligned}$$

For this we need to evaluate the following integral:

$$\begin{aligned} \int_t^s f_2(u) du &= - \int_t^s \left[ \frac{1}{c} + \frac{1}{a}(T-u) \right]^{-1} du \\ &= a \cdot \ln \left[ \frac{1}{c} + \frac{1}{a}(T-u) \right]_{u=t}^{u=s} \\ &= a \cdot \ln \left[ \frac{\frac{1}{c} + \frac{1}{a}(T-s)}{\frac{1}{c} + \frac{1}{a}(T-t)} \right] \end{aligned}$$

Similarly, we have

$$\int_t^s g_2(u) du = b \cdot \ln \left[ \frac{\frac{1}{c} + \frac{1}{b}(T-s)}{\frac{1}{c} + \frac{1}{b}(T-t)} \right]$$

Now we define the integrating factor:

$$\begin{aligned} \mu_f(s) &= \exp \left[ \int_t^s \left( \frac{f_2(u)}{a} - \kappa \right) du \right] \\ &= \frac{\frac{1}{c} + \frac{1}{a}(T-s)}{\frac{1}{c} + \frac{1}{a}(T-t)} \cdot e^{-\kappa(s-t)} \end{aligned}$$

The ODE associated with  $f_{11}$  can be solved using the integrating factor above:

$$\begin{aligned}
& \partial_t f_{11} + \left( \frac{f_2}{a} - \kappa \right) f_{11} = \kappa \\
\Rightarrow & \partial_t f_{11} \cdot \mu_f + \left( \frac{f_2}{a} - \kappa \right) f_{11} \cdot \mu_f = \kappa \cdot \mu_f \\
\Rightarrow & \partial_t [f_{11} \cdot \mu_f] = \kappa \cdot \mu_f \\
\Rightarrow & \underbrace{f_{11}(T)}_{=0} \cdot \underbrace{\mu_f(T)}_{=1} - f_{11}(t) \cdot \underbrace{\mu_f(t)}_{=1} = \int_t^T \kappa \cdot \mu_f(s) ds \\
\Rightarrow & f_{11}(t) = -\kappa \int_t^T \frac{\frac{1}{c} + \frac{1}{a}(T-s)}{\frac{1}{c} + \frac{1}{a}(T-t)} \cdot e^{-\kappa(s-t)} ds
\end{aligned}$$

Finally, we can complete the solution by evaluating the required integral:

$$\begin{aligned}
f_{11}(t) &= -\kappa \int_t^T \frac{\frac{1}{c} + \frac{1}{a}(T-s)}{\frac{1}{c} + \frac{1}{a}(T-t)} \cdot e^{-\kappa(s-t)} ds \\
&= \frac{-\kappa e^{\kappa t}}{\frac{1}{c} + \frac{1}{a}(T-t)} \cdot \int_t^T e^{-\kappa s} \cdot \left[ \frac{1}{c} + \frac{1}{a}(T-s) \right] ds \\
&= \frac{-\kappa e^{\kappa t}}{\frac{1}{c} + \frac{1}{a}(T-t)} \cdot \left[ \left( \frac{1}{c} + \frac{T}{a} \right) \int_t^T e^{\kappa s} ds + \frac{1}{a} \int_t^T s e^{-\kappa s} ds \right] \\
&= \frac{-\kappa e^{\kappa t}}{\frac{1}{c} + \frac{1}{a}(T-t)} \cdot \left[ -\frac{1}{\kappa} \left( \frac{1}{c} + \frac{T}{a} \right) (e^{-\kappa T} - e^{-\kappa t}) + \frac{T}{a\kappa} e^{-\kappa T} - \frac{t}{a\kappa} e^{-\kappa t} + \frac{1}{a\kappa^2} (e^{-\kappa T} - e^{-\kappa t}) \right] \\
&= \frac{e^{\kappa t}}{\frac{1}{c} + \frac{1}{a}(T-t)} \cdot \left[ \left( \frac{1}{c} + \frac{T}{a} \right) (e^{-\kappa T} - e^{-\kappa t}) - \frac{1}{a} (T e^{-\kappa T} - e^{-\kappa t}) + \frac{1}{a\kappa} (e^{-\kappa T} - e^{-\kappa t}) \right] \\
&= -f_2(t) \cdot \left[ \left( \frac{1}{c} + \frac{1}{\kappa a} \right) e^{-\kappa(T-t)} + f_2(t)^{-1} + \frac{1}{\kappa a} \right]
\end{aligned}$$

Similarly, we can set a second integrating factor:

$$\begin{aligned}
\mu_g(s) &= \exp \left[ \int_t^s \left( \frac{g_2(u)}{b} - \kappa \right) du \right] \\
&= \frac{\frac{1}{c} + \frac{1}{b}(T-s)}{\frac{1}{c} + \frac{1}{b}(T-t)} \cdot e^{-\kappa(s-t)}
\end{aligned}$$

and obtain:

$$\begin{aligned}
g_{11}(t) &= \kappa \int_t^T \frac{\frac{1}{c} + \frac{1}{b}(T-s)}{\frac{1}{c} + \frac{1}{b}(T-t)} \cdot e^{-\kappa(s-t)} ds \\
&= g_2(t) \cdot \left[ \left( \frac{1}{c} + \frac{1}{\kappa b} \right) e^{-\kappa(T-t)} + g_2(t)^{-1} + \frac{1}{\kappa b} \right]
\end{aligned}$$